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Journal of Sound and Vibration 272 (2004) 581–606
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Insight into the dynamic condensation technique of non-classically damped models

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Received 23 August 2002; accepted 31 March 2003

Abstract

The dynamic condensation technique fully depends on the definition and computational scheme of the dynamic condensation matrix. Four definitions for the dynamic condensation matrix in the single-mode-, m-mode-, response-dependent dynamic condensation and modal reduction of non-classically damped models are presented. They are, respectively, defined as the relations of the single eigenvector, m eigenvectors, P eigenvectors, and responses between the master and the slave degrees of freedom. Using the complex mode superposition technique, the response-dependent dynamic condensation matrix may be interpreted as any-mode-, including whole-mode, m-mode, P-mode and single-mode, dependent condensation matrix. Computational equations for the dynamic condensation matrix are derived for each of definitions. After the proper introduction of the assumptions for the single-mode and the response-dependent dynamic condensation, the same computational equation is obtained from the former three definitions. In the modal reduction, the dynamic condensation matrix is directly computed from the eigenvector matrix of full model. Because the eigenvector matrix of the non-classically damped models is generally complex, the complex numerical operations are required in the commonly used expression. An alternative expression is derived in which only the real numerical operations are necessary. Furthermore, it is proven that the dynamic condensation matrix and the reduced system matrices resulted from the modal reduction all have real values.

1. Introduction

Modern supercomputers are capable of solving problems involving more than one million equations with one million unknowns. However, they are still not enough to satisfy the needs of

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0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00385-7

some engineers. This phenomenon that the demand of computer storage and speed always exceed existing capabilities has been consistently demonstrated in finite element analysis during the past half-century [1]. In fact, even though the modern supercomputers can handle very large size of engineering problems, the analysis cost is very expensive. Furthermore, the latest supercomputers are usually not available for most of researchers and engineers. These limitations on the hardware (computer storage and speed) and computational costs make computational techniques as important as computer techniques. Efficient computational technique may significantly reduce the computer storage and time required. As we know, the computational effort of finite element analysis is approximately proportional to the cubic of the size of the problem. Therefore, the computational work could be reduced drastically if the size of problem is reduced before the detailed analysis is performed. This is one of the motivations of the development of model reduction techniques.

Dynamic condensation, as an efficient model reduction technique, has been applied to many areas [2]. Many researches on this topic have been done during the past several years [3–17]. These methods proposed are concentrated on undamped models and also valid for proportionally damped models because the proportional damping does not affect the normal modes of undamped models on which most definitions of dynamic condensation matrix are based. However, there are a lot of situations in which the proportional damping assumption is invalid. Examples of such cases are the structures made up of materials with different damping characteristics in different parts, structures equipped with passive (concentrated dampers, etc.) and active control systems, structures with layers of damping materials (smart materials, viscoelastic materials, etc.), and structures with rotating parts (rotor, etc.). The normal modes with real values resulted from the corresponding undamped models can not be used to uncouple the dynamic equations of these non-classically damped models. The state vectors defined in the state space are, hence, commonly used. The size of the resulted system matrices will be doubled automatically compared to those defined in the displacement space. Therefore, the dynamic condensation technique becomes more necessary. Recently, the dynamic condensation technique for non-classically damped systems has received much attention due to the fast applications of the smart or intelligent materials in a large number of engineering structures or systems.

Generally, we may perform model reduction for the non-classically damped models both in the displacement space and in the state space [18]. The simplest approach for non-classically damped models is the extension of Guyan condensation in which Guyan condensation matrix is directly used to reduce the mass, damping, and stiffness matrices of the full model [19,20]. Of course, both the inertia and the damping effects are ignored in this approach. Therefore, the accuracy of the reduced model is usually very low, especially for the models with high damping. The dynamic condensation methods defined in the displacement space, particularly the iterative algorithms for undamped models, may increase the accuracy of the reduced model. However, it is difficult to consider the influence of non-classical damping into the dynamic condensation matrix resulted from these approaches. Error will, hence, be introduced. Although the corresponding reduced model may be convergent, it will not converge to the full model in the interested frequency range [18].

Similar to the exact dynamic condensation for undamped models, its extended version for non-classically damped models may also be derived in the state space using the same logic.

Unfortunately, the resulted dynamic condensation matrix and the system matrices of the reduced model are generally complex. Further analysis of the reduced model is inconvenient and very expensive. Based on the modal reduction method for undamped models, an extended modal reduction method for non-classically damped models was proposed by Kane and Torby [21] and applied to rotor dynamic problems in 1991. The resulted reduced model can preserve exactly all the modes originally selected.

An iterative method for the dynamic condensation of non-classically damped systems was proposed by Qu [2] in 1998. In this method, two governing equations for the dynamic condensation matrix, which relates the eigenvectors associated with the master and the slave degrees of freedom in the state space, were derived. Since the eigenvectors and eigenvalues of the reduced model are not included in the equation, it is unnecessary to solve for the reduced eigenproblem at every iteration. Shortly after, a similar iterative approach was presented by Qu and Chang [22] in which the dynamic condensation matrix is defined as the relation of responses between the master and the slave degrees of freedom.

In 1999, a dynamic condensation approach applicable to non-classically damped structures was proposed by Rivera et al. [23]. This approach is a generalization and extension of the iterative condensation approach for undamped models. In this method, the eigenproperties obtained in an iterative step are used to improve the condensation matrix in the following iterative step.

Based on the extension of the standard subspace iteration method for undamped models, an iterative dynamic condensation method for non-classically damped systems was derived by Qu and Selvam [24]. This method has three advantages. (1) The convergence is much faster than the previous methods, especially when the approximate values of the reduced model are close to the full model. (2) A full proof of the convergence can be made simply. (3) Because there is not any parameter of the reduced model in the governing equation of the dynamic condensation matrix, it is unnecessary to calculate them at each iteration. This makes the iterative scheme much more computationally efficient, especially when the number of the master degrees of freedom is a little large.

The properties of reduced model, accuracy for example, resulted from the dynamic condensation depend on the full model and the dynamic condensation matrix. Generally, the full model is given and unchangeable during the condensation. Therefore, the reduced model fully depends on the dynamic condensation matrix. Different definitions and computational schemes may result in different reduced models with different accuracy and purposes.

Four definitions for the dynamic condensation matrix of non-classically damped models are to be presented. They are single-mode-, *m*-mode-, response-dependent dynamic condensation and modal reduction, respectively. The relation of these definitions will be explained. Computational equations of the dynamic condensation matrix are derived from these definitions respectively. Because the eigenvector matrix of non-classically damped model defined in the state space is complex, the complex numerical operation is required in the modal reduction approach. The question whether the dynamic condensation matrix as well as the reduced system matrices is complex or not has not been answered up to now. In this paper, it will be proven that the dynamic condensation matrix resulted from this method has real values. Furthermore, an alternative computation is provided to avoid the complex numerical computation.

2. Theory of non-classically damped model

The dynamic equations of a viscously damped system can be expressed in a matrix form as

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{X}}(t) + \mathbf{K}\mathbf{X}(t) = \mathbf{f}(t), \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K} \in R^{n\times n}$ are the mass, damping, and stiffness matrices of the damped model. They are assumed to be symmetric. $\mathbf{f}(t) \in R^n$ is an external force vector. $\mathbf{X}(t)$, $\dot{\mathbf{X}}(t)$, and $\ddot{\mathbf{X}}(t) \in R^n$ are the displacement, velocity, and acceleration response vectors of the system. They are functions of time t. In the dynamic condensation, the total degrees of freedom of the full model are divided into the master degrees of freedom, denoted by m, and the slave degrees of freedom, denoted by s. With this division, Eq. (1) can be rewritten in a partitioned form as

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{X}}_m(t) \\ \ddot{\mathbf{X}}_s(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{C}_{mm} & \mathbf{C}_{ms} \\ \mathbf{C}_{sm} & \mathbf{C}_{ss} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{X}}_m(t) \\ \dot{\mathbf{X}}_s(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{X}_m(t) \\ \mathbf{X}_s(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_m(t) \\ \mathbf{f}_s(t) \end{Bmatrix}, \tag{2}$$

where $\mathbf{M}_{mm}$, $\mathbf{C}_{mm}$, $\mathbf{K}_{mm} \in R^{m\times m}$; $\mathbf{M}_{ms}$, $\mathbf{C}_{ms}$, $\mathbf{K}_{ms} \in R^{m\times s}$; $\mathbf{M}_{sm}$, $\mathbf{C}_{sm}$, $\mathbf{K}_{sm} \in R^{s\times m}$; $\mathbf{M}_{ss}$, $\mathbf{C}_{ss}$, $\mathbf{K}_{ss} \in R^{s\times s}$; $\ddot{\mathbf{X}}_m(t)$, $\dot{\mathbf{X}}_m(t)$, $\mathbf{X}_m(t)$, $\mathbf{f}_m(t) \in R^m$; $\ddot{\mathbf{X}}_s(t)$, $\dot{\mathbf{X}}_s(t)$, $\mathbf{X}_s(t)$, $\mathbf{f}_s(t) \in R^s$.

Due to the non-classical damping, Eq. (1) is very difficult to be uncoupled in the displacement space. The state space formulation is generally introduced. The dynamic equation in the state space are generally given by

$$\mathbf{A}\mathbf{Y}(t) - \mathbf{B}\dot{\mathbf{Y}}(t) = \mathbf{F}(t), \tag{3}$$

where the system matrix $\mathbf{A}, \mathbf{B} \in R^{2n\times 2n}$, the state vector $\mathbf{Y}(t) \in R^{2n}$, and the force vector $\mathbf{F}(t) \in R^{2n}$ are defined as

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \quad \mathbf{B} = -\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \quad \mathbf{Y}(t) = \begin{Bmatrix} \mathbf{X}(t) \\ \dot{\mathbf{X}}(t) \end{Bmatrix}, \quad \mathbf{F}(t) = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{Bmatrix} \tag{4}$$

or

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \quad \mathbf{Y}(t) = \begin{Bmatrix} \mathbf{X}(t) \\ \dot{\mathbf{X}}(t) \end{Bmatrix}, \quad \mathbf{F}(t) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}(t) \end{Bmatrix}. \tag{5}$$

It can be seen that the size of matrices in the state space is automatically doubled.

The eigenproblem corresponding to Eq. (3) may be expressed as

$$(\mathbf{A} - \lambda\mathbf{B})\tilde{\boldsymbol{\psi}} = \mathbf{0} \tag{6}$$

in which λ and $\tilde{\boldsymbol{\psi}} \in C^{2n}$ are the eigenvalue (or complex frequency) and eigenvector of the full model. Generally, this eigenproblem has $2n$ eigenvalues and eigenvectors. They appear in n complex conjugate pairs. The compact form for all the eigenpairs may be expressed as

$$\mathbf{A}\tilde{\boldsymbol{\Psi}} = \mathbf{B}\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Omega}}. \tag{7}$$

The complex conjugate eigenvector matrix $\tilde{\boldsymbol{\Psi}} \in C^{2n\times 2n}$ and eigenvalue matrix $\tilde{\boldsymbol{\Omega}} \in C^{2n\times 2n}$ have the forms:

$$\tilde{\boldsymbol{\Psi}} = \begin{bmatrix} \boldsymbol{\Psi} & \boldsymbol{\Psi}^* \\ \boldsymbol{\Psi}\boldsymbol{\Omega} & \boldsymbol{\Psi}^*\boldsymbol{\Omega}^* \end{bmatrix} = [\tilde{\boldsymbol{\psi}}_1\tilde{\boldsymbol{\psi}}_2\cdots\tilde{\boldsymbol{\psi}}_n\tilde{\boldsymbol{\psi}}_1^*\tilde{\boldsymbol{\psi}}_2^*\cdots\tilde{\boldsymbol{\psi}}_n^*], \quad \tilde{\boldsymbol{\Omega}} = \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}^* \end{bmatrix}. \tag{8}$$

The superscript "*" denotes the complex conjugation. The orthogonalities are given by

$$\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{A}\tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Omega}}, \quad \tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{B}\tilde{\mathbf{\Psi}} = \mathbf{I}, \tag{9}$$

where $\mathbf{I} \in R^{2n\times 2n}$ is an identity matrix.

3. Dynamic condensation

As aforementioned, the total degrees of freedom of a full model are divided into the master degrees of freedom and the salve degrees of freedom in the dynamic condensation. The former will be retained in the reduced model and the latter will be deleted from the full model and they are also referred to as the kept and deleted degrees of freedom. It is well known that the selection of master degrees of freedom has influence on the speed of convergence or the accuracy of the reduced model in the physical type dynamic condensation [2] which will be shown using a numerical example. However, a full discussion on what and how many degrees of freedom should be chosen as the master degrees of freedom is beyond the scope of this paper.

3.1. Single-mode-dependent dynamic condensation

In the single-mode-dependent dynamic condensation, the condensation matrix is defined as

$$\tilde{\boldsymbol{\psi}}_s = \mathbf{R}_{\mathrm{I}}\tilde{\boldsymbol{\psi}}_m. \tag{10}$$

$\tilde{\boldsymbol{\psi}}_m$ and $\tilde{\boldsymbol{\psi}}_s$ are the subvectors of the eigenvector $\tilde{\boldsymbol{\psi}}$ and will be defined later. $\mathbf{R}_{\mathrm{I}}$ is referred to as the single-mode-dependent dynamic condensation matrix. Its physical meaning is the relation of an eigenvector between the master and slave degrees of freedom. This eigenvector could be any eigenvector of the full model. Different eigenvectors may have different dynamic condensation matrices. Consequently, this dynamic condensation matrix is single-mode dependent.

It is a little difficult to compute the dynamic condensation matrix directly from Eq. (10) even though the eigenvector is available. Therefore, some auxiliary equations may be required. For this problem, Eq. (6) is usually selected as an auxiliary equation.

With the above arrangement of the total degrees of freedom, Eq. (6) may be partitioned as

$$\left(\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{ms} \\ \mathbf{A}_{sm} & \mathbf{A}_{ss} \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{B}_{mm} & \mathbf{B}_{ms} \\ \mathbf{B}_{sm} & \mathbf{B}_{ss} \end{bmatrix}\right) \begin{Bmatrix} \tilde{\boldsymbol{\psi}}_m \\ \tilde{\boldsymbol{\psi}}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{11}$$

If the system matrices $\mathbf{A}$ and $\mathbf{B}$ defined in Eq. (5) are used, the submatrices in Eq. (11) are given by

$$\mathbf{A}_{mm} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_{mm} \\ \mathbf{K}_{mm} & \mathbf{C}_{mm} \end{bmatrix}, \quad \mathbf{A}_{ms} = \mathbf{A}_{sm}^{\mathrm{T}} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms} & \mathbf{C}_{ms} \end{bmatrix}, \quad \mathbf{A}_{ss} = \begin{bmatrix} \mathbf{0} & \mathbf{K}_{ss} \\ \mathbf{K}_{ss} & \mathbf{C}_{ss} \end{bmatrix}, \tag{12a}$$

$$\mathbf{B}_{mm} = \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{mm} \end{bmatrix}, \quad \mathbf{B}_{ms} = \mathbf{B}_{sm}^{\mathrm{T}} = \begin{bmatrix} \mathbf{K}_{ms} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ms} \end{bmatrix}, \quad \mathbf{B}_{ss} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ss} \end{bmatrix}. \tag{12b}$$

The submatrices corresponding to the system matrices $\mathbf{A}$ and $\mathbf{B}$ defined in Eq. (5) may be similarly obtained.A simple multiplication of the matrices on the left-hand side of Eq. (11) expands the equation into two equations, namely,

$$(\mathbf{A}_{mm} - \lambda\mathbf{B}_{mm})\tilde{\boldsymbol{\psi}}_m + (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})\tilde{\boldsymbol{\psi}}_s = \mathbf{0}, \tag{13a}$$

$$(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm})\tilde{\boldsymbol{\psi}}_m + (\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})\tilde{\boldsymbol{\psi}}_s = \mathbf{0}. \tag{13b}$$

The relation of the eigenvector between the master and the slave degrees of freedom may be obtained from Eq. (13b) as

$$\tilde{\boldsymbol{\psi}}_s = -(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm})\tilde{\boldsymbol{\psi}}_m. \tag{14}$$

According to the definition, the computational expression for the dynamic condensation matrix may be obtained as

$$\mathbf{R}_{\mathrm{I}}(\lambda) = -(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}). \tag{15}$$

Clearly, the dynamic condensation matrix $\mathbf{R}_{\mathrm{I}}$ is a function of the eigenvalue of full model. Generally, the eigenvalue is unknown. Furthermore, different eigenvalues (modes) lead to different dynamic condensation matrix. Therefore, this matrix is single-mode dependent. Of course, if other auxiliary equations are introduced, different computational expressions may be derived.

When the dynamic condensation matrix is available, it is interested to find the system matrices for the reduced model. Substituting Eq. (14) into Eq. (13a) produces

$$\mathbf{D}_R(\lambda)\tilde{\boldsymbol{\psi}}_m = \mathbf{0} \tag{16}$$

in which the dynamic stiffness matrix of the reduced model is given by

$$\mathbf{D}_R(\lambda) = (\mathbf{A}_{mm} - \lambda\mathbf{B}_{mm}) - (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}). \tag{17}$$

Similar to the full model in the state space, two system matrices $\mathbf{A}_R$ and $\mathbf{B}_R$ of the reduced model are introduced and they satisfy

$$\mathbf{D}_R(\lambda) = \mathbf{A}_R - \lambda\mathbf{B}_R. \tag{18}$$

The two reduced system matrices may be computed from $\mathbf{D}_R(\lambda)$ as

$$\mathbf{B}_R(\lambda) = -\frac{\mathrm{d}\mathbf{D}_R(\lambda)}{\mathrm{d}\lambda}, \quad \mathbf{A}_R(\lambda) = \mathbf{D}_R(\lambda) - \lambda\frac{\mathrm{d}\mathbf{D}_R(\lambda)}{\mathrm{d}\lambda}. \tag{19}$$

The relations of the reduced system matrices used in Eq. (19) are very similar to the Leung's theorem [25] which was proposed for undamped models. Introducing Eq. (17) into Eq. (19), the reduced system matrices may be obtained as

$$\mathbf{A}_R(\lambda) = \mathbf{A}_{mm} + \mathbf{R}_{\mathrm{I}}(\lambda)^{\mathrm{T}}\mathbf{A}_{sm} + \mathbf{A}_{ms}\mathbf{R}_{\mathrm{I}}(\lambda) + \mathbf{R}_{\mathrm{I}}(\lambda)^{\mathrm{T}}\mathbf{A}_{ss}\mathbf{R}_{\mathrm{I}}(\lambda), \tag{20a}$$

$$\mathbf{B}_R(\lambda) = \mathbf{B}_{mm} + \mathbf{R}_{\mathrm{I}}(\lambda)^{\mathrm{T}}\mathbf{B}_{sm} + \mathbf{B}_{ms}\mathbf{R}_{\mathrm{I}}(\lambda) + \mathbf{R}_{\mathrm{I}}(\lambda)^{\mathrm{T}}\mathbf{B}_{ss}\mathbf{R}_{\mathrm{I}}(\lambda). \tag{20b}$$

The details of this derivation can be found in Appendix B.

Using the dynamic condensation matrix in Eq. (10), the whole eigenvector may be expressed as

$$\tilde{\boldsymbol{\psi}} = \begin{bmatrix} \tilde{\boldsymbol{\psi}}_m \\ \tilde{\boldsymbol{\psi}}_s \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{R}_{\mathrm{I}}(\lambda) \end{bmatrix}\tilde{\boldsymbol{\psi}}_m = \mathbf{T}_{\mathrm{I}}(\lambda)\tilde{\boldsymbol{\psi}}_m \tag{21}$$

in which $\mathbf{T}_{\mathrm{I}}(\lambda)$ is referred to as the co-ordinate transformation matrix or reduced basis. Introducing Eq. (21) into Eq. (6) or (11) and premultiplying it by the transpose of the matrix $\mathbf{T}_{\mathrm{I}}(\lambda)$ leads to

$$(\mathbf{A}_R - \lambda\mathbf{B}_R)\tilde{\boldsymbol{\psi}}_m = \mathbf{0}. \tag{22}$$

The reduced system matrices can be written in terms of the co-ordinate transformation matrix as

$$\mathbf{A}_R(\lambda) = \mathbf{T}_{\mathrm{I}}(\lambda)^{\mathrm{T}}\mathbf{A}\mathbf{T}_{\mathrm{I}}(\lambda), \quad \mathbf{B}_R(\lambda) = \mathbf{T}_{\mathrm{I}}(\lambda)^{\mathrm{T}}\mathbf{B}\mathbf{T}_{\mathrm{I}}(\lambda). \tag{23}$$

Substituting Eq. (21) into Eq. (23), we arrive at the same definition of the reduced system matrices as those in Eq. (20). This shows that the reduced system matrices resulted from the direct substitution and the co-ordinate transformations are the same. In fact, this is generally right when the dynamic condensation matrix exactly represents the relation of the eigenvector between the master and the slave degrees of freedom. Researches show that the reduced model computed from the co-ordinate transformation has higher accuracy than that from direct substitution if the dynamic condensation matrix is approximate.

3.2. *m*-Mode-dependent dynamic condensation

In the *m*-mode-dependent dynamic condensation, the dynamic condensation matrix is defined as

$$\tilde{\boldsymbol{\Psi}}_{sm} = \mathbf{R}_{\mathrm{II}}\tilde{\boldsymbol{\Psi}}_{mm} \tag{24}$$

$\tilde{\boldsymbol{\Psi}}_{mm} \in C^{2m\times 2m}$ and $\tilde{\boldsymbol{\Psi}}_{sm} \in C^{2s\times 2m}$ are the submatrices of eigenvector matrix $\tilde{\boldsymbol{\Psi}}_m$ and will be defined later. Matrix $\mathbf{R}_{\mathrm{II}}$ is referred to as the *m*-mode-dependent dynamic condensation matrix. The physical meaning is the relations of the *m* eigenvectors between the master and the slave degrees of freedom. The *m* eigenvectors may be in the lowest frequency range or any frequency range of the full model. Whenever the *m* eigenvectors change, the dynamic condensation matrix $\mathbf{R}_{\mathrm{II}}$ changes. Consequently, this dynamic condensation matrix is *m*-mode dependent. Because all the *m* modes have the same dynamic condensation matrix, it is very efficient and convenient to use the reduced model resulted from the dynamic condensation matrix $\mathbf{R}_{\mathrm{II}}$. This definition has been frequently used in Refs. [2,18,23,24].

Clearly, if the *m* pairs of modes are available, the dynamic condensation matrix could be determined directly from Eq. (24) as

$$\mathbf{R}_{\mathrm{II}} = \tilde{\boldsymbol{\Psi}}_{sm}\tilde{\boldsymbol{\Psi}}_{mm}^{-1}. \tag{25}$$

Although this computational expression is quite simple, the eigenproblem analysis of the full model in the state space should be performed before the dynamic condensation. Furthermore, the complex numerical operations are required because of the complex nature of the eigenvectors. It will be proven later that the dynamic condensation matrix computed from Eq. (25) is real although the complex matrices are used.

Another way to find the computational expression for the dynamic condensation matrix is based on the auxiliary Eq. (7) in which only *m* modes are considered, that is,

$$\mathbf{A}\tilde{\boldsymbol{\Psi}}_m = \mathbf{B}\tilde{\boldsymbol{\Psi}}_m\tilde{\boldsymbol{\Omega}}_{mm}. \tag{26}$$

The m modes used in Eq. (26) actually include their complex conjugated modes. Thus, the eigenvector matrix $\tilde{\mathbf{\Psi}}_m \in C^{2n\times 2m}$ and eigenvalue matrix $\tilde{\mathbf{\Omega}}_{mm} \in C^{2m\times 2m}$ are defined as

$$\tilde{\mathbf{\Psi}}_m = \begin{bmatrix} \mathbf{\Psi}_m & \mathbf{\Psi}_m^* \\ \mathbf{\Psi}_m\mathbf{\Omega}_{mm} & \mathbf{\Psi}_m^*\mathbf{\Omega}_{mm}^* \end{bmatrix}, \quad \tilde{\mathbf{\Omega}}_{mm} = \begin{bmatrix} \mathbf{\Omega}_{mm} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}_{mm}^* \end{bmatrix}. \tag{27}$$

Due to the complex conjugated nature, they are still referred to as m modes for simplicity. This definition is used in whole paper.

With the same arrangement of the total degrees of freedom, Eq. (26) may be partitioned as

$$\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{ms} \\ \mathbf{A}_{sm} & \mathbf{A}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{mm} & \mathbf{B}_{ms} \\ \mathbf{B}_{sm} & \mathbf{B}_{ss} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm} \end{bmatrix} \tilde{\mathbf{\Omega}}_{mm}. \tag{28}$$

The eigenvector matrix may be written in terms of the dynamic condensation matrix defined in Eq. (24) as

$$\tilde{\mathbf{\Psi}}_m = \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mm} \\ \tilde{\mathbf{\Psi}}_{sm} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{R}_{\mathrm{II}} \end{bmatrix} \tilde{\mathbf{\Psi}}_{mm} = \mathbf{T}_{\mathrm{II}} \tilde{\mathbf{\Psi}}_{mm}. \tag{29}$$

Using the co-ordinate transformation matrix $\mathbf{T}_{\mathrm{II}}$ or the dynamic condensation matrix $\mathbf{R}_{\mathrm{II}}$, the reduced system matrices may be similarly obtained as shown in Eq. (20) or (23). Finally, the eigenproblem of the reduced model is given by

$$\mathbf{A}_R \tilde{\mathbf{\Psi}}_{mm} = \mathbf{B}_R \tilde{\mathbf{\Psi}}_{mm} \tilde{\mathbf{\Omega}}_{mm}. \tag{30}$$

The second equation of Eq. (28) is rewritten as

$$\mathbf{A}_{sm}\tilde{\mathbf{\Psi}}_{mm} + \mathbf{A}_{ss}\tilde{\mathbf{\Psi}}_{sm} = \mathbf{B}_{sm}\tilde{\mathbf{\Psi}}_{mm}\tilde{\mathbf{\Omega}}_{mm} + \mathbf{B}_{ss}\tilde{\mathbf{\Psi}}_{sm}\tilde{\mathbf{\Omega}}_{mm} \tag{31}$$

which leads to

$$\tilde{\mathbf{\Psi}}_{sm} = \mathbf{A}_{ss}^{-1}(\mathbf{B}_{sm}\tilde{\mathbf{\Psi}}_{mm}\tilde{\mathbf{\Omega}}_{mm} + \mathbf{B}_{ss}\tilde{\mathbf{\Psi}}_{sm}\tilde{\mathbf{\Omega}}_{mm} - \mathbf{A}_{sm}\tilde{\mathbf{\Psi}}_{mm}). \tag{32}$$

Introducing the definition equation of the m-mode-dependent dynamic condensation matrix, shown in Eq. (24), into Eq. (32) and post-multiplying it by the inverse of the eigenvector matrix $\tilde{\mathbf{\Psi}}_{mm}$ results in

$$\mathbf{R}_{\mathrm{II}} = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}_{\mathrm{II}})\tilde{\mathbf{\Psi}}_{mm}\tilde{\mathbf{\Omega}}_{mm}\tilde{\mathbf{\Psi}}_{mm}^{-1} - \mathbf{A}_{sm}\right\rfloor. \tag{33}$$

Rewriting Eq. (30) gives

$$\mathbf{B}_R^{-1}\mathbf{A}_R = \tilde{\mathbf{\Psi}}_{mm}\tilde{\mathbf{\Omega}}_{mm}\tilde{\mathbf{\Psi}}_{mm}^{-1}. \tag{34}$$

Substituting Eq. (34) into Eq. (33), the governing equation for the dynamic condensation matrix $\mathbf{R}_{\mathrm{II}}$ may be obtained as

$$\mathbf{R}_{\mathrm{II}} = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}_{\mathrm{II}})\mathbf{B}_R^{-1}\mathbf{A}_R - \mathbf{A}_{sm}\right\rfloor, \tag{35}$$

when Eq. (7) is considered as an auxiliary equation. If other equations are selected as the auxiliary equations, other computational expressions for the m-mode-dependent dynamic condensation matrix may be derived [2,24].

3.3. Response-dependent dynamic condensation

In the response-dependent dynamic condensation, the dynamic condensation matrix is defined as the relation of the responses between the master and the slave degrees of freedom, that is,

$$\mathbf{Y}_s(t) = \mathbf{R}_{\text{III}}\mathbf{Y}_m(t). \tag{36}$$

The matrix $\mathbf{R}_{\text{III}}$ is referred to as the response-dependent dynamic condensation matrix. This definition was used by Qu and Change [22] in 2000.

The dynamic condensation matrix may possibly be obtained directly from this definition. In this procedure, the system responses are simulated using any accurate time integration scheme. The displacement response vector $\mathbf{Y}(t)$ at all the degrees of freedom is then sampled at a series of different times during the simulations. The dynamic condensation matrix is then computed from these sampled response vectors. When the number of the sampled vectors approach to infinite, the dynamic condensation matrix $\mathbf{R}_{\text{III}}$ will converge to the exact. In practice, only a limit number of samples are used and error will, hence, be introduced. To improve the features of reduced model, these sampled vectors should be orthogonalized. To our knowledge if the procedure is performed in the displacement space, the reduced model will have better features. This research is under way.

Also, the computational expression for the dynamic condensation matrix may be derived from the auxiliary Eq. (3). Using the same division of the total degrees of freedom, Eq. (3) can be partitioned as

$$\begin{bmatrix} \mathbf{A}_{mm} & \mathbf{A}_{ms} \\ \mathbf{A}_{sm} & \mathbf{A}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{Y}_m(t) \\ \mathbf{Y}_s(t) \end{Bmatrix} - \begin{bmatrix} \mathbf{B}_{mm} & \mathbf{B}_{ms} \\ \mathbf{B}_{sm} & \mathbf{B}_{ss} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{Y}}_m(t) \\ \dot{\mathbf{Y}}_s(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_m(t) \\ \mathbf{F}_s(t) \end{Bmatrix}. \tag{37}$$

For the force vector defined in Eq. (5), the subvectors are given by

$$\mathbf{F}_m(t) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_m(t) \end{Bmatrix}, \quad \mathbf{F}_s(t) = \begin{Bmatrix} \mathbf{0} \\ \mathbf{f}_s(t) \end{Bmatrix}. \tag{38}$$

Using the dynamic condensation matrix defined in Eq. (36), the state vector $\mathbf{Y}(t)$ may be expressed as

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{Y}_m(t) \\ \mathbf{Y}_s(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{R}_{\text{III}} \end{bmatrix} \mathbf{Y}_m(t) = \mathbf{T}_{\text{III}}\mathbf{Y}_m(t). \tag{39}$$

Differentiating both sides of Eq. (39) with respect to time leads to

$$\dot{\mathbf{Y}}(t) = \mathbf{T}_{\text{III}}\dot{\mathbf{Y}}_m(t). \tag{40}$$

Introducing Eqs. (39) and (40) into Eq. (37) and premultiplying it by the transpose of matrix $\mathbf{T}_{\text{III}}$ gives

$$\mathbf{A}_R\mathbf{Y}_m(t) - \mathbf{B}_R\dot{\mathbf{Y}}_m(t) = \mathbf{F}_R(t). \tag{41}$$

In which the equivalent force vector acting on the reduced model is given by

$$\mathbf{F}_R(t) = \mathbf{F}_m(t) + \mathbf{R}_{\text{III}}^{\text{T}}\mathbf{F}_s(t) \tag{42}$$

and reduced system matrices may be computed similarly from Eq. (20) or (23).

After setting $\mathbf{F}_s(t) = \mathbf{0}$ in Eq. (37), its second equation may be rewritten as

$$\mathbf{A}_{sm}\mathbf{Y}_m(t) + \mathbf{A}_{ss}\mathbf{Y}_s(t) = \mathbf{B}_{sm}\dot{\mathbf{Y}}_m(t) + \mathbf{B}_{ss}\dot{\mathbf{Y}}_s(t). \tag{43}$$

For the harmonic response $\mathbf{Y}(t) = \mathbf{Y}_0 e^{st}$, one has

$$\dot{\mathbf{Y}}_m(t) = s\mathbf{Y}_m(t), \dot{\mathbf{Y}}_s(t) = s\mathbf{Y}_s(t). \tag{44}$$

Substituting Eq. (44) into Eq. (43) and rearranging it results in

$$\mathbf{Y}_s(t) = -(\mathbf{A}_{ss} - s\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - s\mathbf{B}_{sm})\mathbf{Y}_m(t). \tag{45}$$

According to the definition of the dynamic condensation matrix in Eq. (36), we have

$$\mathbf{R}_{\mathrm{III}} = -(\mathbf{A}_{ss} - s\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - s\mathbf{B}_{sm}). \tag{46}$$

Eq. (46) is the computational expression for the response-dependent dynamic condensation matrix. This equation is different from Eq. (15). The harmonic frequency in Eq. (46) is prescribed and the eigenvalue (or complex frequency) λ in Eq. (15) is unknown. Clearly, the dynamic condensation matrix computed from Eq. (46) is only exact at the harmonic frequency s. If the frequency changes, the dynamic condensation matrix has to be recomputed. Therefore, this expression is only utilized for the harmonically excited models. The more general form to compute the response-dependent dynamic condensation matrix will be shown in the next section.

3.4. Modal reduction

In the modal reduction, the dynamic condensation matrix is defined as

$$\tilde{\mathbf{\Psi}}_{sP} = \mathbf{R}_{\mathrm{IV}}\tilde{\mathbf{\Psi}}_{mP}. \tag{47}$$

Researches show that when the number of the modes P is equal to or less than the number of the master degrees of freedom, the resulted reduced model is physically usable. Once the P modes of the full model are computed, the dynamic condensation matrix $\mathbf{R}_{\mathrm{IV}}$ may be directly computed from Eq. (47). Because the matrix $\tilde{\mathbf{\Psi}}_{mP}$ is generally not a square matrix, the generalized inverse is required to obtain the condensation matrix, that is,

$$\mathbf{R}_{\mathrm{IV}} = \tilde{\mathbf{\Psi}}_{sP}\tilde{\mathbf{\Psi}}^{+}_{mP} \tag{48}$$

in which the generalized inverse $\tilde{\mathbf{\Psi}}^{+}_{mP}$ is given by

$$\tilde{\mathbf{\Psi}}^{+}_{mP} = (\tilde{\mathbf{\Psi}}^{\mathrm{T}}_{mP}\tilde{\mathbf{\Psi}}_{mP})^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}_{mP}. \tag{49}$$

After the dynamic condensation matrix is computed, the co-ordinate transformation matrices may be defined as

$$\mathbf{T}_{\mathrm{IV}} = \begin{bmatrix} \mathbf{I} \\ \mathbf{R}_{\mathrm{IV}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \tilde{\mathbf{\Psi}}_{sP}\tilde{\mathbf{\Psi}}^{+}_{mP} \end{bmatrix} \tag{50}$$

or

$$\mathbf{T}_{\mathrm{IV}} = \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mP}\tilde{\mathbf{\Psi}}^{+}_{mP} \\ \tilde{\mathbf{\Psi}}_{sP}\tilde{\mathbf{\Psi}}^{+}_{mP} \end{bmatrix}. \tag{51}$$

Finally, the reduced system matrices are given by

$$\mathbf{A}_R = \mathbf{T}_{\mathrm{IV}}^{\mathrm{T}}\mathbf{A}\mathbf{T}_{\mathrm{IV}}, \quad \mathbf{B}_{\mathrm{IV}} = \mathbf{T}_{\mathrm{IV}}^{\mathrm{T}}\mathbf{B}\mathbf{T}_{\mathrm{IV}}. \tag{52}$$

Most importantly, the modal reduction method will produce a reduced model which exactly preserves the selected modes of the full model. Therefore, it has been widely used in the test-analysis model correlation. However, the reduced model resulted from this scheme is generally rank deficient. This leads to many difficulties in further dynamic analyses performed on the reduced model.

4. Discussions on the dynamic condensation matrix

4.1. Dynamic condensation matrix

Four definitions for the dynamic condensation matrix have been presented in the preceding section. Because $\mathbf{R}_{\mathrm{I}}(\lambda)$ is single-mode dependent, it is a function of the unknown eigenvalue. The definitions of $\mathbf{R}_{\mathrm{II}}$ and $\mathbf{R}_{\mathrm{IV}}$ look very similar. The only difference is the number of modes included in the definition. In the former the number is fixed as m, while it can be any number between 1 and m in the latter.

According to the mode superposition theory, the displacements in time domain depend on whole modes of the full model. Therefore, the response-dependent dynamic condensation matrix $\mathbf{R}_{\mathrm{III}}$ is whole-mode dependent. Using the complex mode superposition, the displacement vector $\mathbf{Y}(t)$ can be expressed in terms of the eigenvector matrix and the associated modal co-ordinate $\mathbf{q}(t)$, namely,

$$\mathbf{Y}(t) = \tilde{\mathbf{\Psi}}\mathbf{q}(t). \tag{53}$$

Actually, it is unnecessary and impossible to include all modes in the mode superposition for a large size of model. Hence, mode truncation is usually used. If p modes are included in the mode superposition, Eq. (53) is rewritten as

$$\mathbf{Y}(t) = \tilde{\mathbf{\Psi}}_p\mathbf{q}_p(t). \tag{54}$$

Its partitioned form is given by

$$\mathbf{Y}(t) = \begin{Bmatrix} \mathbf{Y}_m(t) \\ \mathbf{Y}_s(t) \end{Bmatrix} = \begin{bmatrix} \tilde{\mathbf{\Psi}}_{mp} \\ \tilde{\mathbf{\Psi}}_{sp} \end{bmatrix}\mathbf{q}_p(t). \tag{55}$$

Introducing Eq. (55) into Eq. (36) gives

$$\tilde{\mathbf{\Psi}}_{sp}\mathbf{q}_p(t) = \mathbf{R}_{\mathrm{III}}\tilde{\mathbf{\Psi}}_{mp}\mathbf{q}_p(t), \tag{56}$$

which leads to

$$\tilde{\mathbf{\Psi}}_{sp} = \mathbf{R}_{\mathrm{III}}\tilde{\mathbf{\Psi}}_{mp}. \tag{57}$$

The p can be any integer from 1 to n. Hence, the dynamic condensation matrix $\mathbf{R}_{\mathrm{III}}$ is any-mode dependent if the mode truncation is applied. Because the reduced model has a maximum of m degrees of freedom, the maximum p is actually m if the reduced model is used to define the relation

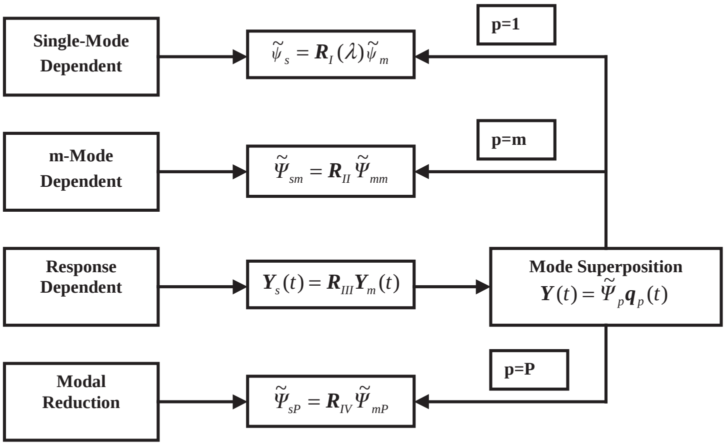

Fig. 1. Relations of the definitions of dynamic condensation matrix.

between $\dot{\mathbf{Y}}_m(t)$ and $\mathbf{Y}_m(t)$ as will be shown in Eq. (69). The relation of the different definitions for the dynamic condensation matrices is shown in Fig. 1.

4.2. Computational expressions for condensation matrices

Although the single-mode-dependent dynamic condensation matrix $\mathbf{R}_\mathrm{I}(\lambda)$ is expressed exactly and explicitly in Eq. (15), it is a function of the unknown eigenvalue λ. Iterative scheme is usually necessary to estimate the dynamic condensation matrix. The inverse of the dynamic stiffness matrix or its equivalence is required at each iteration. If another mode is interested, the whole computation has to be repeated again. Consequently, this computational expression is very computationally expensive.

The flexible matrix in Eq. (15) may be expressed as

$$(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1} = \mathbf{A}_{ss}^{-1} + \lambda\mathbf{A}_{ss}^{-1}\mathbf{B}_{ss}(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}. \tag{58}$$

Introducing Eq. (58) into Eq. (15) gives

$$\mathbf{R}_\mathrm{I}(\lambda) = -\mathbf{A}_{ss}^{-1}\mathbf{A}_{sm} + \lambda\mathbf{A}_{ss}^{-1}\mathbf{B}_{sm} - \lambda\mathbf{A}_{ss}^{-1}\mathbf{B}_{ss}(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}). \tag{59}$$

Substituting Eq. (15) into the right-hand side of Eq. (59) and rearranging its results in

$$\mathbf{R}_\mathrm{I}(\lambda) = \mathbf{A}_{ss}^{-1}[\lambda(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}_\mathrm{I}(\lambda)) - \mathbf{A}_{sm}]. \tag{60}$$

Eq. (60) is exactly equivalent to Eq. (15). Clearly, the expensive computation of matrix $(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}$ or its equivalence is avoided in Eq. (60). Unfortunately, the dynamic condensation matrix $\mathbf{R}_\mathrm{I}(\lambda)$ is included on the right-hand side of this equation. Iterative scheme is, hence, required to compute the matrix.

Post-multiplying both sides of Eq. (60) by the eigenvector at the master degrees of freedom results in

$$\mathbf{R}_\mathrm{I}(\lambda)\tilde{\boldsymbol{\psi}}_m = \mathbf{A}_{ss}^{-1}[\lambda(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}_\mathrm{I}(\lambda))\tilde{\boldsymbol{\psi}}_m - \mathbf{A}_{sm}\tilde{\boldsymbol{\psi}}_m]. \tag{61}$$

The eigenproblem of the reduced model shown in Eq. (22) may be expressed as

$$\mathbf{A}_R\tilde{\boldsymbol{\psi}}_m = \lambda\mathbf{B}_R\tilde{\boldsymbol{\psi}}_m. \tag{62}$$

Premultiplying both sides of Eq. (62) by the inverse of the system matrix $\mathbf{B}_R$ gives

$$\mathbf{B}_R^{-1}\mathbf{A}_R\tilde{\boldsymbol{\psi}}_m = \lambda\tilde{\boldsymbol{\psi}}_m. \tag{63}$$

Because the reduced model has m degrees of freedom, it has a maximum of m modes. Any of the m modes satisfies Eq. (63). Substituting $\mathbf{B}_R^{-1}\mathbf{A}_R\tilde{\boldsymbol{\psi}}_m$ for $\lambda\tilde{\boldsymbol{\psi}}_m$ on the right-hand side of Eq. (61) leads to

$$\mathbf{R}_\mathrm{I}(\lambda)\tilde{\boldsymbol{\psi}}_m = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}_\mathrm{I}(\lambda))\mathbf{B}_R^{-1}\mathbf{A}_R - \mathbf{A}_{sm}\right\rfloor\tilde{\boldsymbol{\psi}}_m. \tag{64}$$

Eq. (64) is still single-mode dependent due to $\tilde{\boldsymbol{\psi}}_m$. Because it is valid for any of the m modes, it may be directly rewritten in terms of the m modes as

$$\hat{\mathbf{R}}_\mathrm{I}\tilde{\boldsymbol{\Psi}}_{mm} = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\hat{\mathbf{R}}_\mathrm{I})\mathbf{B}_R^{-1}\mathbf{A}_R - \mathbf{A}_{sm}\right\rfloor\tilde{\boldsymbol{\Psi}}_{mm}. \tag{65}$$

Since the eigenvector matrix $\tilde{\boldsymbol{\Psi}}_{mm}$ is generally full-ranked square matrix, Eq. (65) leads to

$$\hat{\mathbf{R}}_\mathrm{I} = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\hat{\mathbf{R}}_\mathrm{I})\mathbf{B}_R^{-1}\mathbf{A}_R - \mathbf{A}_{sm}\right\rfloor. \tag{66}$$

Eq. (66) is the governing equation of the dynamic condensation matrix $\hat{\mathbf{R}}_\mathrm{I}$. Clearly, it is not single-mode dependent any more due to the use of the relation in Eq. (62). This condensation matrix is actually m-mode dependent as shown in Eq. (35). The relation of the computational expressions for the single-mode- and the m-mode-dependent dynamic condensation matrices is shown in Fig. 2.

Researches show that the convergence of Eq. (15) or (60) is usually faster than Eq. (66) when proper iterative scheme is implemented. However, if more than one modes are interested, Eq. (15) or (60) may have to be re-calculated repeatedly for each mode which leads to expensive computation. The m modes may be computed simultaneously when Eq. (66) is implemented to reduce the full model. If r modes which are less than m modes are interested, this equation is still valid. For this case, the convergence of the r modes rather than the m modes need to be checked.

For the single-mode-dependent dynamic condensation, different modes have different reduced system matrices which usually incur difficulties in further dynamic analysis. There is only one reduced model for all the m modes interested in the m-mode-dependent dynamic condensation.

For harmonic responses, the computational equation of the response-dependent dynamic condensation matrix was presented in Eq. (46). Unfortunately, the harmonic response is generally very rare in reality. For a general response, the Fourier transformation may be used to transfer it into a summation or integration of a series of harmonic responses. However, different harmonic responses may have different frequency s which leads to repeated calculation of the dynamic condensation matrix. Therefore, Eq. (46) is only used when the dynamic condensation matrix at one specific frequency s is interested. Since s is generally known, iterative scheme is unnecessary.

From Eq. (43) one has

$$\mathbf{Y}_s(t) = \mathbf{A}_{ss}^{-1}(\mathbf{B}_{sm}\dot{\mathbf{Y}}_m(t) + \mathbf{B}_{ss}\dot{\mathbf{Y}}_s(t) - \mathbf{A}_{sm}\mathbf{Y}_m(t)). \tag{67}$$

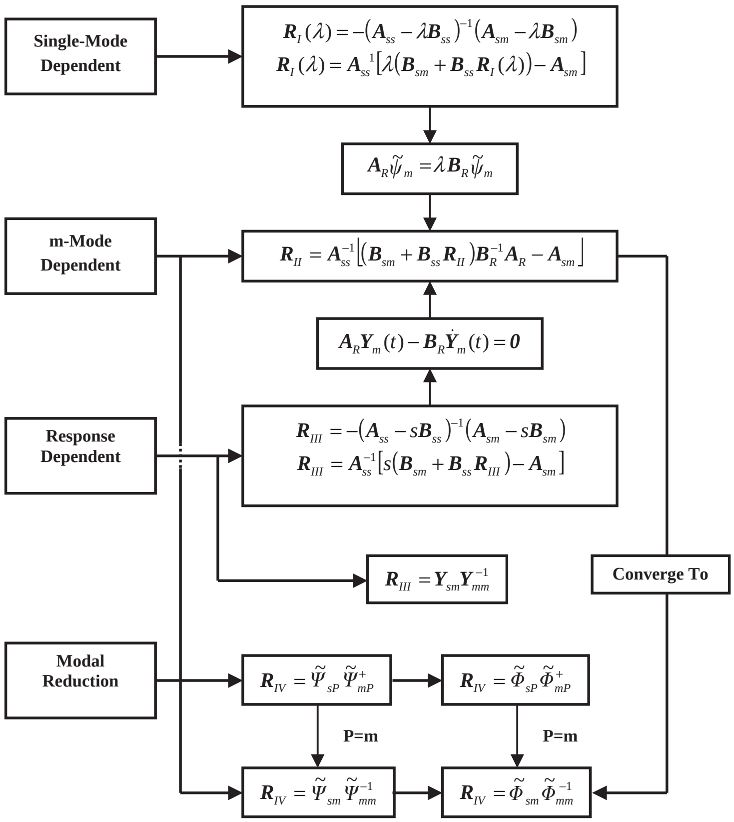

Fig. 2. Relations of the computational equations for dynamic condensation matrix.

Introducing the definition of the dynamic condensation matrix $\mathbf{R}_{\mathrm{III}}$ in Eq. (36) into both sides of Eq. (67) gives

$$\mathbf{R}_{\mathrm{III}}\mathbf{Y}_m(t) = \mathbf{A}_{ss}^{-1}[(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}_{\mathrm{III}})\dot{\mathbf{Y}}_m(t) - \mathbf{A}_{sm}\mathbf{Y}_m(t)]. \tag{68}$$

Eq. (68) is still response dependent.

The free vibration corresponding to the forced vibration of the reduced model shown in Eq. (41) is given by

$$\mathbf{A}_R\mathbf{Y}_m(t) - \mathbf{B}_R\dot{\mathbf{Y}}_m(t) = \mathbf{0}. \tag{69}$$

From which one has

$$\dot{\mathbf{Y}}_m(t) = \mathbf{B}_R^{-1}\mathbf{A}_R\mathbf{Y}_m(t). \tag{70}$$

Because the reduced model only has m degrees of freedom, mode truncated errors are included in the responses computed from Eq. (69). Therefore, the relation of the responses shown in Eq. (70) is approximate. Introducing Eq. (70) into Eq. (68) results in

$$\hat{\mathbf{R}}_{\mathrm{III}}\mathbf{Y}_m(t) = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\hat{\mathbf{R}}_{\mathrm{III}})\mathbf{B}_R^{-1}\mathbf{A}_R - \mathbf{A}_{sm}\right\rfloor\mathbf{Y}_m(t). \tag{71}$$

Due to the time-dependent term on both sides, the governing equation of the dynamic condensation matrix $\hat{\mathbf{R}}_{\rm III}$ may be obtained from Eq. (71) as

$$\mathbf{R}_{\rm III} = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}_{\rm III})\mathbf{B}_R^{-1}\mathbf{A}_R - \mathbf{A}_{sm}\right\rfloor. \tag{72}$$

Because of the application of Eq. (70) or (69), the governing equation reduces from response (whole-mode) dependent to m-mode dependent. The relation of these computational equations for the dynamic condensation matrices is shown in Fig. 2.

From the viewpoint of the vibration modes, the definition of $\mathbf{R}_{\rm III}$ requires the condensation matrix representing the relation of the all eigenvectors simultaneously. Condensation matrices $\mathbf{R}_{\rm II}$ and $\mathbf{R}_{\rm I}$, on the other hand, only represent the relation of m eigenvectors and single eigenvector, respectively. The requirement in the definition of $\mathbf{R}_{\rm III}$ is generally too strict to be satisfied for practical problems. Therefore, the definition of $\mathbf{R}_{\rm III}$ is stricter than that of $\mathbf{R}_{\rm II}$ which is stricter than $\mathbf{R}_{\rm I}$. The use of relations in Eqs. (62) and (69), respectively, strengthens and weakens the requirement of the definitions of $\mathbf{R}_{\rm I}$ and $\mathbf{R}_{\rm III}$. Therefore, the same governing equations are obtained for $\mathbf{R}_{\rm I}$, $\mathbf{R}_{\rm II}$, and $\mathbf{R}_{\rm III}$ when these relations are used.

These three governing equations are all implicit. Hence, iterative scheme is usually used to solve them. The commonly used iterative scheme is given in Refs. [2,22,23]

$$\begin{cases} \mathbf{R}^{(0)} = -\mathbf{A}_{ss}^{-1}\mathbf{A}_{sm} \\ \mathbf{R}^{(i)} = \mathbf{A}_{ss}^{-1}\left\lfloor(\mathbf{B}_{sm} + \mathbf{B}_{ss}\mathbf{R}^{(i-1)})(\mathbf{B}_R^{(i-1)})^{-1}\mathbf{A}_R^{(i-1)} - \mathbf{A}_{sm}\right\rfloor \end{cases} \tag{73}$$

in which $i = 1, 2, \cdots$. The $(i-1)$th approximate system matrices $\mathbf{A}_R^{(i-1)}$ and $\mathbf{B}_R^{(i-1)}$ of the reduced model may be computed similarly from Eq. (20). Because the matrices $\mathbf{A}$ and $\mathbf{B}$ are all real, the resulted dynamic condensation matrices $\mathbf{R}$ ($\mathbf{R}_{\rm I}$, $\mathbf{R}_{\rm II}$, and $\mathbf{R}_{\rm III}$) are also real. Consequently, the reduced system matrices are all real.

5. Conversion of complex operation into real operation

It can be seen from the computational expression of the dynamic condensation matrix in Eq. (48) that the numerical operation for the complex values is required because the eigenvector matrices are generally complex. To reduce the unnecessary computational effort, an alternative approach, in which only the real numerical operation is required, will be provided in the following.

As we known that the complex eigenvector matrix $\tilde{\mathbf{\Psi}}$ has the form of

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{\Psi}^* \\ \mathbf{\Psi\Omega} & \mathbf{\Psi}^*\mathbf{\Omega}^* \end{bmatrix} = [\,\hat{\mathbf{\Psi}} \quad \hat{\mathbf{\Psi}}^*\,], \tag{74}$$

where $\hat{\mathbf{\Psi}} \in C^{2n\times n}$ is given by

$$\hat{\mathbf{\Psi}} = \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{\Psi\Omega} \end{bmatrix}. \tag{75}$$

The complex matrix $\hat{\mathbf{\Psi}}$ can be expressed in terms of two real matrices $\mathbf{U}$ and $\mathbf{V} \in R^{2n \times n}$ as

$$\hat{\mathbf{\Psi}} = \mathbf{U} + \mathrm{i}\mathbf{V}. \tag{76}$$

Hence, the conjugate matrix $\hat{\mathbf{\Psi}}^*$ is given by

$$\hat{\mathbf{\Psi}}^* = \mathbf{U} - \mathrm{i}\mathbf{V} \tag{77}$$

and the whole eigenvector is expressed as

$$\tilde{\mathbf{\Psi}} = [\, \hat{\mathbf{\Psi}} \quad \hat{\mathbf{\Psi}}^* \,] = [\, \mathbf{U} + \mathrm{i}\mathbf{V} \quad \mathbf{U} - \mathrm{i}\mathbf{V} \,]. \tag{78}$$

If only the P modes are considered, Eq. (78) may be rewritten as

$$\tilde{\mathbf{\Psi}}_P = [\, \hat{\mathbf{\Psi}}_P \quad \hat{\mathbf{\Psi}}_P^* \,] = [\, \mathbf{U}_P + \mathrm{i}\mathbf{V}_P \quad \mathbf{U}_P - \mathrm{i}\mathbf{V}_P \,]. \tag{79}$$

Define the unitary transformation $\mathbf{J} \in C^{2P \times 2P}$ as

$$\mathbf{J} = \frac{1}{2} \begin{bmatrix} \mathbf{J}_- & \mathbf{J}_+ \\ \mathbf{J}_+ & \mathbf{J}_- \end{bmatrix} \tag{80}$$

in which $\mathbf{J}_-$ and $\mathbf{J}_+ \in C^{P \times P}$ are diagonal matrices and defined as

$$\mathbf{J}_- = \mathbf{I} - \mathrm{i}\mathbf{I}, \quad \mathbf{J}_+ = \mathbf{I} + \mathrm{i}\mathbf{I}. \tag{81}$$

$\mathbf{I} \in R^{P \times P}$ is an identity matrix. Clearly, the inverse of the $\mathbf{J}$ matrix is given by

$$\mathbf{J}^{-1} = \frac{1}{2} \begin{bmatrix} \mathbf{J}_+ & \mathbf{J}_- \\ \mathbf{J}_- & \mathbf{J}_+ \end{bmatrix}. \tag{82}$$

Define matrix $\tilde{\mathbf{\Phi}}_P$ with the order of $2n \times 2P$ as

$$\tilde{\mathbf{\Phi}}_P = \tilde{\mathbf{\Psi}}_P \mathbf{J}^{-1}. \tag{83}$$

Introducing Eqs. (79) and (82) into Eq. (83) gives

$$\tilde{\mathbf{\Phi}}_P = [\, \mathbf{U}_P - \mathbf{V}_P \quad \mathbf{U}_P + \mathbf{V}_P \,]. \tag{84}$$

Clearly, $\tilde{\mathbf{\Phi}}_P$ is a real matrix. From Eq. (83), the complex conjugate eigenvector matrix may be expressed as the product of a real matrix and a unitary transformation matrix, namely,

$$\tilde{\mathbf{\Psi}}_P = \tilde{\mathbf{\Phi}}_P \mathbf{J}. \tag{85}$$

If the same division of the total degrees of freedom is applied to Eq. (85), one has

$$\tilde{\mathbf{\Psi}}_{mP} = \tilde{\mathbf{\Phi}}_{mP} \mathbf{J} \tag{86a}$$

$$\tilde{\mathbf{\Psi}}_{sP} = \tilde{\mathbf{\Phi}}_{sP} \mathbf{J}. \tag{86b}$$

Substituting Eq. (86) into Eq. (49) leads to

$$\tilde{\mathbf{\Psi}}_{mP}^+ = (\tilde{\mathbf{\Psi}}_{mP}^{\mathrm{T}} \tilde{\mathbf{\Psi}}_{mP})^{-1} \tilde{\mathbf{\Psi}}_{mP}^{\mathrm{T}} = \mathbf{J}^{-1} (\tilde{\mathbf{\Phi}}_{mP}^{\mathrm{T}} \tilde{\mathbf{\Phi}}_{mP})^{-1} \tilde{\mathbf{\Phi}}_{mP}^{\mathrm{T}} = \mathbf{J}^{-1} \tilde{\mathbf{\Phi}}_{mP}^+. \tag{87}$$

Hence, the dynamic condensation matrix is given by

$$\mathbf{R}_{\mathrm{IV}} = \tilde{\mathbf{\Psi}}_{sP} \tilde{\mathbf{\Psi}}_{mP}^+ = \tilde{\mathbf{\Phi}}_{sP} \tilde{\mathbf{\Phi}}_{mP}^+. \tag{88}$$

Clearly, the dynamic condensation matrix $\mathbf{R}_{IV}$ is real. Similarly, it is easy to prove that the co-ordinate transformation matrix in Eq. (51) is also real. Consequently the system matrices of the reduced model are all real.

6. Numerical example

Two iterative methods will be considered to demonstrate the features of the dynamic condensation in the state space. They have the same form of governing equation as shown in Eq. (35) while the system matrices are, respectively, defined by Eqs (4) and (5). For simplicity, they are referred to as Method I and Method II.

A floating raft isolation system is utilized in the numerical example. The details of the isolation may refer Ref. [22]. The finite element method is utilized to discretize the raft and the base. The former has 24 rectangular elements, 35 nodes, and 105 degrees of freedom and the latter has 14 rectangular elements, 24 nodes, and 72 degrees of freedom. Therefore, the isolation system has totally 179 degrees of freedom. Due to the non-classically viscous damping included, the state space formulation has to be used in the dynamic condensation.

The lowest ten complex frequencies of the floating raft isolation system calculated from the full model are listed in Table 1 and are considered as the exact values for comparison purpose. The equivalent modal frequencies and damping ratios of the complex frequencies are also given in Table 1. Suppose that the complex frequency has the form:

$$\lambda_j = -\alpha_j \pm \mathrm{i}\beta_j \quad (j = 1, 2, \ldots, m). \tag{89}$$

The modal frequency and damping ratio are given by

$$\omega_j = \sqrt{\alpha_j^2 + \beta_j^2}, \quad \xi_j = \alpha_j/\omega_j \quad (j = 1, 2, \ldots, m). \tag{90}$$

Two cases for the selection of master degrees of freedom are considered. In case I the degrees of freedom associated with the two machines and the translational degrees of freedom at nodes 2, 4,

Table 1
Complex frequencies, damping ratios, and modal frequencies

Mode	Complex frequency		Damping ratio	Modal frequency
	Real part	Imaginary part		
1	−0.33428	27.4784	0.01216	27.4804
2	−0.45405	30.7270	0.01478	30.7303
3	−0.32173	67.4671	0.00477	67.4679
4	−6.46483	226.7590	0.02850	226.8512
5	−9.76748	227.9016	0.04282	228.1108
6	−6.72764	238.6425	0.02818	238.7373
7	−7.49290	335.9309	0.02230	336.0144
8	−12.94492	418.3195	0.03093	418.5197
9	−17.88711	533.9339	0.03348	534.2334
10	−16.40820	542.8270	0.03021	543.0750

8, 9, 14, 22 in raft and at nodes 6, 21 in base are selected as the master degrees of freedom. For case II, the degrees of freedom associated with the two machines and the translational degrees of freedom at nodes 1, 7, 15, 17, 19, 21, 29, 35 in raft and at nodes 7, 8, 9, 10, 16 in base are selected. Thus, the reduced models have 10 and 15 degrees of freedom in the two cases. As we known, the higher modes usually converge much slower than the lower modes. Thus, only the higher five modes, that is, the sixth through the tenth modes, are considered. The absolute relative errors of the sixth through tenth complex frequencies are plotted in Figs. 3–7. In these figures, A, B, C, and D are defined in Table 2.

Generally, the modal frequencies approach to those of the full model with the iterations. The convergence is very fast at the first several iterations and becomes slower and slower with the increase of the number of iterations. The modal frequencies computed from the reduced model are higher than those of full model. This means that the reduced model approaches to the full model from above and that the reduced model is stiffer than the full model. Although the damping ratio generally converges, it is not as clear as the modal frequency. Therefore, the modal frequency is usually used as the major factor and the damping ratio as a minor factor in the convergent check.

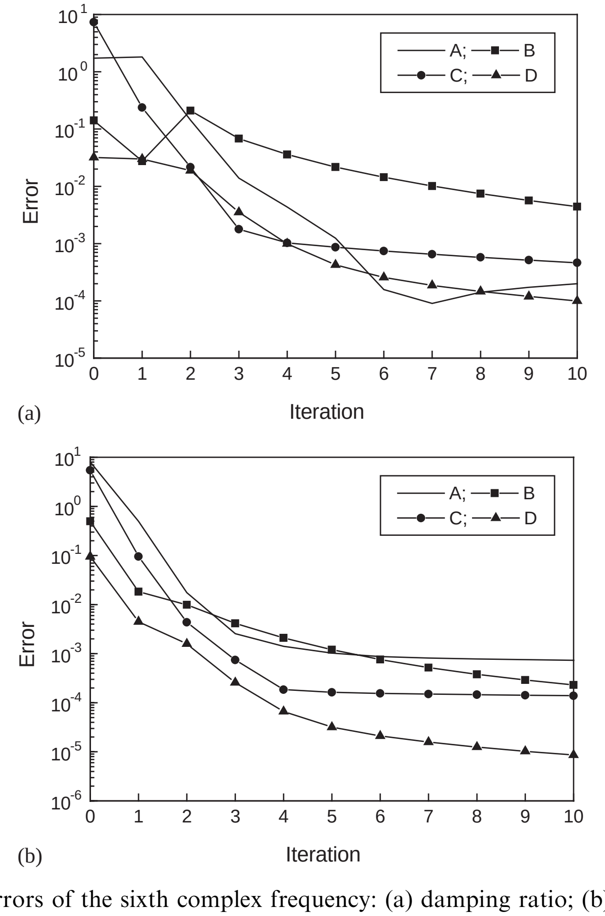

Fig. 3. Absolute errors of the sixth complex frequency: (a) damping ratio; (b) modal frequency.

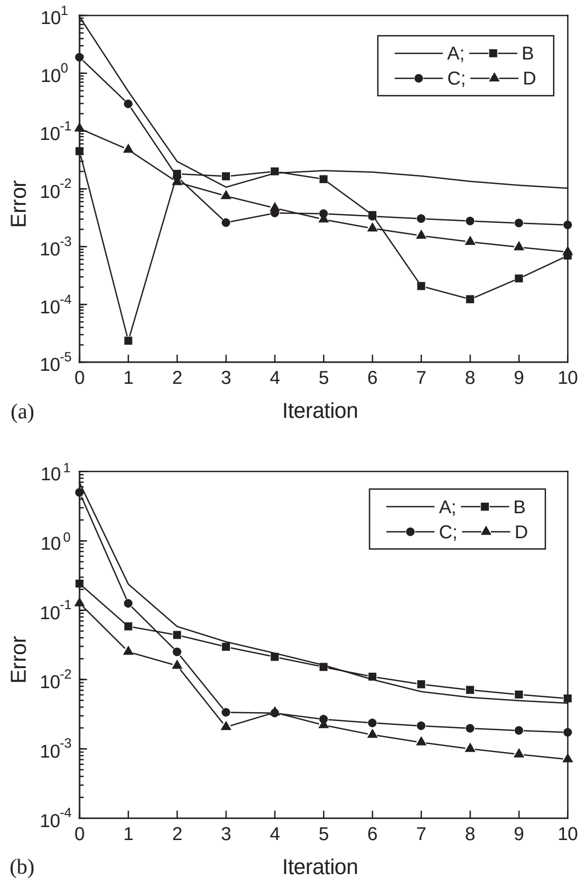

Fig. 4. Absolute errors of the seventh complex frequency: (a) damping ratio; (b) modal frequency.

The errors resulted from the reduced model with 15 degrees of freedom are much smaller than those from the reduced model with 10 degrees of freedom, especially for the higher order of modal frequencies. This means that the convergence will become everlasting faster with the increase of the number of master degrees of freedom. However, a big number of master degrees of freedom will lead to more computational work. Consequently, how many degrees of freedom should be selected as the master degrees of freedom is problem-dependent. Usually, the ratio of the number to the number of modes interested is a value between 1.5 and 2.

7. Conclusions

The definitions of the single-mode-, m-mode-, response-dependent dynamic condensation matrices and modal reduction matrix have been proposed. The corresponding computational equations and the relationship of these definitions were presented. For the modal reduction in the

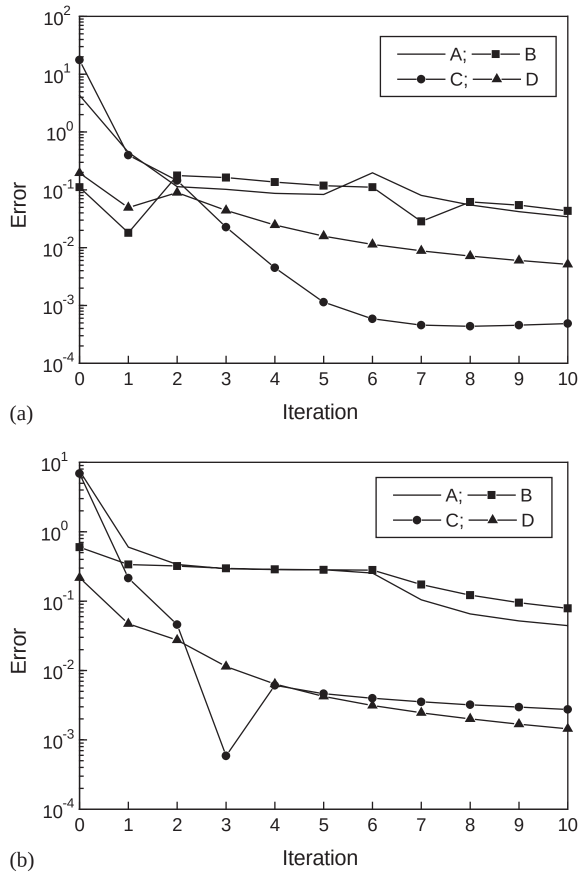

Fig. 5. Absolute errors of the eighth complex frequency: (a) damping ratio; (b) modal frequency.

state space, an alternative expression has been derived to avoid the complex numerical computations.

The single-mode-dependent dynamic condensation matrix is defined as the relation of single eigenvector between the master and the slave degrees of freedom. Based on the definition and the auxiliary Eq. (6), a computational expression was derived. It is a function of the unknown eigenvalue although this expression is exact. If the mode changes, the dynamic condensation matrix has to be calculated again. One disadvantage of this expression is that the very expensive inverse process or its equivalence is required at each mode considered. Furthermore, the resulted reduced model is also eigenvalue dependent.

The m-mode-dependent dynamic condensation matrix is defined as the relation of the m eigenvectors between the master and the slave degrees of freedom. These m modes are generally in the lowest frequency range of full model. They also could be in any frequency range. Only one reduced model is defined by the m modes. Therefore, all the m modes may be simultaneously computed from this reduced model. Furthermore, it is very convenient and efficient to perform further analyses on this reduced model. The dynamic condensation matrix may directly compute from the definition equation if the m modes are available. This leads to a specific case of the modal

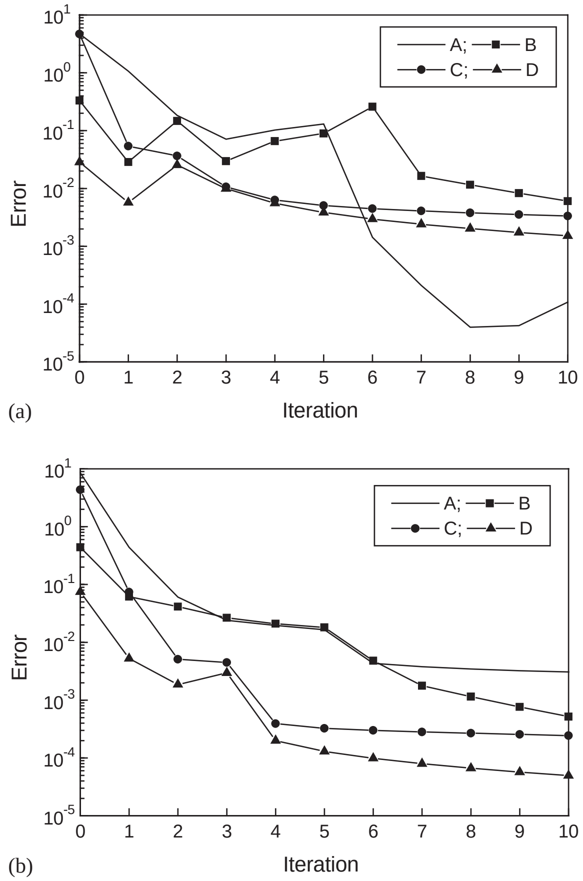

Fig. 6. Absolute errors of the ninth complex frequency: (a) damping ratio; (b) modal frequency.

reduction. By using different auxiliary equations, different computational expressions for the dynamic condensation matrix may be derived.

The response-dependent dynamic condensation matrix is defined as the relation of the responses between the master and the slave degrees of freedom. The dynamic condensation matrix may be directly computed from this definition. In the procedure, the sampled response vectors at a series of different times are required. Using the mode superposition method, the response-dependent dynamic condensation matrix may be interpreted as any-mode-, including whole-mode, m-mode and single-mode-dependent matrix. For the harmonic responses, an expression for computing the dynamic condensation matrix was provided. However, it is very computational expensive.

After the application of the relations in Eqs. (62) and (69) for the single-mode-dependent and the response-dependent dynamic condensation matrices, respectively, one new governing equation has been derived for each of the dynamic condensation matrices. They are same as that for the m-mode-dependent dynamic condensation matrix. Since the computational equation is implicit, iterative scheme is usually required to compute the dynamic condensation matrix. The fact that the dynamic condensation matrix is a real matrix is clearly shown in the new equations. Because

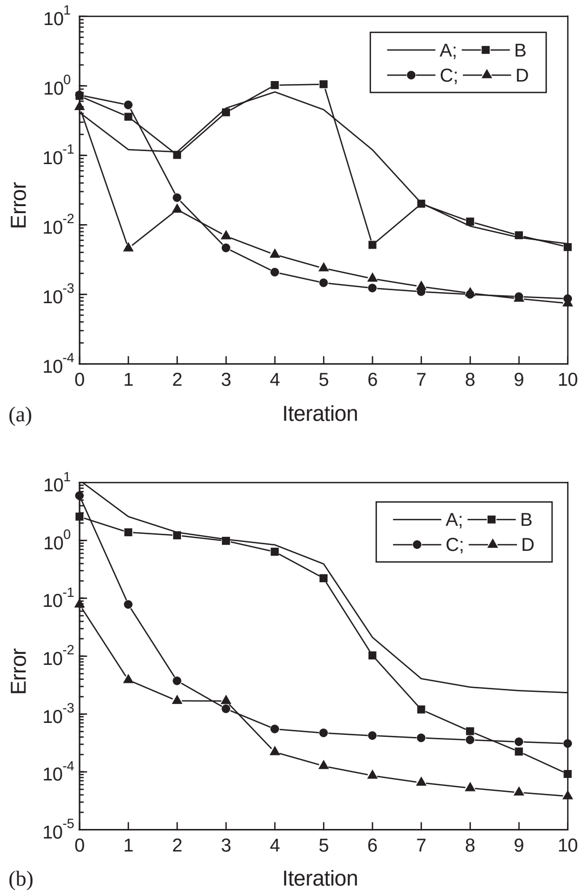

Fig. 7. Absolute errors of the tenth complex frequency: (a) damping ratio; (b) modal frequency.

Table 2
Four cases considered

Case	Method	Masters (number)
A	I	I (10)
B	II	I (10)
C	I	II (15)
D	II	II (15)

of the application of Eq. (62), the resulted dynamic condensation matrix is not single-mode-dependent any more but m-mode dependent. Similarly, Eq. (69) makes the resulted dynamic condensation matrix m-mode dependent rather than response dependent.

The dynamic condensation matrix in the modal reduction is actually any-mode dependent. For practical purpose, the number of modes considered should be less than or equal to the number of master degrees of freedom. In this reduction, the dynamic condensation matrix is directly computed from the eigenvector matrix of full model. Because the eigenvector matrix of the non-classically damped models is generally complex, the complex operation is required in the commonly used expression. An alternative expression has been derived in which only the real numerical operation is required. It has been proven that the dynamic condensation matrix and the reduced system matrices resulted from the modal reduction all have real values.

The reduced model computed from the dynamic condensation are defined in the subspace of the full space. Therefore, each co-ordinate has its physical meaning. Since the reduced model can represent the full model in the interested frequency range, any dynamic analyses that are valid for the full model could be performed on the reduced model. The dynamic condensation technique may be performed on the structural level and the substructural level. Efficient schemes for the model reduction of non-classically damped models may be derived by combining the dynamic condensation technique and the dynamic substructure technique. Such research is under way.

Appendix A. Nomenclature

$\mathbf{A}$	$(2n \times 2n)$ system matrix of full model in the state space defined in Eqs. (4) and (5)
$\mathbf{A}_R$	$(2m \times 2m)$ system matrix of reduced model in the state space defined in Eqs. (20) and (23)
$\mathbf{B}$	$(2n \times 2n)$ system matrix of full model in the state space defined in Eqs. (4) and (5)
$\mathbf{B}_R$	$(2m \times 2m)$ system matrix of reduced model in the state space defined in Eq. (20) or (23)
$\mathbf{C}$	$(n \times n)$ damping matrix of full model
$\mathbf{f}$	$(n \times 1)$ force vector in the displacement space
$\mathbf{F}_R$	$(2m \times 1)$ force vector of reduced model in the state space defined in Eq. (42)
$\mathbf{F}$	$(2n \times 1)$ force vector of full model in the state space defined in Eqs. (4) and (5)
$\mathbf{I}$	$(2m \times 2m$ or $2n \times 2n$, or $2P \times 2P)$ identity matrix
$\mathbf{J}$	$(2P \times 2P)$ complex unitary transformation matrix defined in Eq. (80)
$\mathbf{K}$	$(n \times n)$ stiffness matrix of full model
$\mathbf{M}$	$(n \times n)$ mass matrix of full model
m	Number of master degrees of freedom
n	Number of the total degrees of freedom
$\mathbf{q}(t)$	$(2n \times 1)$ modal co-ordinates
$\mathbf{R}_{\mathrm{I}}$	$(2s \times 2m)$ single-mode-dependent dynamic condensation matrix defined in Eq. (10)
$\mathbf{R}_{\mathrm{II}}$	$(2s \times 2m)$ m-mode-dependent dynamic condensation matrix defined in Eq. (24)
$\mathbf{R}_{\mathrm{III}}$	$(2s \times 2m)$ response-dependent dynamic condensation matrix defined in Eq. (36)
$\mathbf{R}_{\mathrm{IV}}$	$(2s \times 2m)$ modal reduction matrix defined in Eq. (47)
s	Complex response frequency used in Eq. (44)
	Number of slave degrees of freedom
t	Time
$\mathbf{T}_{\mathrm{I}}$	$(2n \times 2m)$ co-ordinate transformation matrix of single-mode-dependent dynamic condensation defined in Eq. (21)

$\mathbf{T}_{II}$ — $(2n \times 2m)$ co-ordinate transformation matrix of m-mode-dependent dynamic condensation defined in Eq. (29)
$\mathbf{T}_{III}$ — $(2n \times 2m)$ co-ordinate transformation matrix of response-dependent dynamic condensation defined in Eq. (39)
$\mathbf{T}_{IV}$ — $(2n \times 2m)$ co-ordinate transformation matrix of any-mode-dependent dynamic condensation defined in Eqs. (50) or (51)
$\mathbf{U}$ — $(2n \times n)$ real part of the complex eigenvector matrix $\hat{\mathbf{\Psi}}$ defined in Eq. (76)
$\mathbf{V}$ — $(2n \times n)$ imaginary part of the complex eigenvector matrix $\hat{\mathbf{\Psi}}$ defined in Eq. (76)
$\mathbf{X}$ — $(n \times 1)$ displacement response in the displacement space
$\dot{\mathbf{X}}$ — $(n \times 1)$ velocity response in the displacement space
$\ddot{\mathbf{X}}$ — $(n \times 1)$ acceleration response in the displacement space
$\mathbf{Y}$ — $(2n \times 1)$ state vector defined in Eq. (4) or (5)
$\tilde{\mathbf{\Phi}}_P$ — $(2n \times 2P)$ real matrix defined in Eqs. (83) and (84)
λ — complex frequency or eigenvalue defined in Eq. (6)
$\tilde{\mathbf{\Omega}}_{mm}$ — $(2m \times 2m)$ eigenvalue matrix in the state space defined in Eq. (26)
$\tilde{\mathbf{\Omega}}$ — $(2n \times 2n)$ eigenvalue matrix of full model in the state space defined in Eq. (7)
$\tilde{\boldsymbol{\psi}}$ — $(2n \times 1)$ eigenvector of full model defined in Eq. (6)
$\tilde{\mathbf{\Psi}}_m$ — $(2n \times 2m)$ eigenvector matrix of full model in the state space defined in Eq. (26)
$\hat{\mathbf{\Psi}}$ — $(2n \times n)$ eigenvector matrix of full model in the state space defined in Eq. (75)
$\tilde{\mathbf{\Psi}}$ — $(2n \times 2n)$ eigenvector matrix of full model in the state space defined in Eq. (7)

Subscript
m — Number of master degrees of freedom or eigenvectors
Variable associated with the master degrees of freedom
p — Number of eigenvectors or modal co-ordinates
P — Number of eigenvectors or modal co-ordinates
s — Number of slave degrees of freedom or eigenvectors
variable associated with the slave degrees of freedom

Superscript
0 — Initial approximation
i, $i-1$ — The ith or $(i-1)$th approximation
T — Transpose
* — Complex conjugation
$+$ — Generalized inverse

Appendix B. Derivation of Eq. (20)

The derivative of the dynamic system matrix $\mathbf{D}_R(\lambda)$ with respect to the eigenvalue λ is given by

$$\begin{aligned}\frac{\mathrm{d}\mathbf{D}_R(\lambda)}{\mathrm{d}\lambda} = & -\mathbf{B}_{mm} + \mathbf{B}_{ms}(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}) \\ & - (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}\mathbf{B}_{ss}(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}) \\ & + (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}\mathbf{B}_{sm}. \qquad (B.1)\end{aligned}$$

Using the dynamic condensation matrix given in Eqs. (15) and (B.1) may be simplified as

$$\frac{d\mathbf{D}_R(\lambda)}{d\lambda} = -\mathbf{B}_{mm} - \mathbf{B}_{ms}\mathbf{R}_\mathrm{I} - \mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{B}_{ss}\mathbf{R}_\mathrm{I} - \mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{B}_{sm}. \tag{B.2}$$

Introducing Eq. (B.2) into the first equation of Eq. (19), the reduced system matrix $\mathbf{B}_R$ can be obtained as shown in Eq. (20b).The dynamic system matrix of the reduced model may be rewritten as

$$\begin{aligned}\mathbf{D}_R(\lambda) = {} & (\mathbf{A}_{mm} - \lambda\mathbf{B}_{mm}) - (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}) \\ & + (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}) \\ & - (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})^{-1}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm})\end{aligned} \tag{B.3}$$

or in a concise form as

$$\mathbf{D}_R(\lambda) = (\mathbf{A}_{mm} - \lambda\mathbf{B}_{mm}) + (\mathbf{A}_{ms} - \lambda\mathbf{B}_{ms})\mathbf{R}_\mathrm{I} + \mathbf{R}_\mathrm{I}^\mathrm{T}(\mathbf{A}_{ss} - \lambda\mathbf{B}_{ss})\mathbf{R}_\mathrm{I} + \mathbf{R}_\mathrm{I}^\mathrm{T}(\mathbf{A}_{sm} - \lambda\mathbf{B}_{sm}) \tag{B.4}$$

Rearranging the items on the right-hand side of Eq. (B.4) gives

$$\mathbf{D}_R(\lambda) = \mathbf{A}_{mm} + \mathbf{A}_{ms}\mathbf{R}_\mathrm{I} + \mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{A}_{ss}\mathbf{R}_\mathrm{I} + \mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{A}_{sm} - \lambda(\mathbf{B}_{mm} + \mathbf{B}_{ms}\mathbf{R}_\mathrm{I} + \mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{B}_{ss}\mathbf{R}_\mathrm{I} + \mathbf{R}_\mathrm{I}^\mathrm{T}\mathbf{B}_{sm}) \tag{B.5}$$

Substituting Eqs. (B.2) and (B.5) into the second equation of Eq. (19), the reduced system matrix $\mathbf{A}_R$ is obtained as shown in Eq. (20a).

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